Constructing Lagrangian submanifolds

★ <u>Symplectic reduction</u>: $G \circlearrowright (M,\omega)$ Ham. action, $c \in Z(\mathfrak{g}^*)$ regular value of μ

$M_{red} = \mu^{-1}(c)/G$, ω_{red} st. $\pi^*\omega_{red} = \omega_{|\mu^{-1}(c)}$.

<u>Prop:</u> $L_{red} \subset (M_{red}, \omega_{red})$ Lagrangian $\Rightarrow \pi^{-1}(L_{red}) \subset (M,\omega)$ Lagrangian.

<u>Direct proof:</u> $\mu^{-1}(c) \xrightarrow{\pi} M_{red}$ submersion (locally trivial fibration, fiber $\simeq G$)
so $L = \pi^{-1}(L_{red})$ is a smooth submanifold of $\dim = \dim(L_{red}) + \dim G$
$= \frac{1}{2}\dim M$ ✓
and $v_i \in TL \Rightarrow \bar{v}_i = d\pi(v_i) \in TL_{red}$
so $\omega(v_1,v_2) = \pi^*\omega_{red}(v_1,v_2) = \omega_{red}(\bar{v}_1,\bar{v}_2) \underset{\text{Lag.}}{=} 0$ ✓.

(More generally: $L_1 \subset (M_1,\omega_1)$ Lagrangian, $L_{12} \subset (M_1^- \times M_2, -\omega_1 \oplus \omega_2)$ Lag. correspondence, under suitable assumptions $L_{12} \circ L_1$ Lag. $\subset M_2$. See HW 1 Problem 2.
Apply to $\Gamma_\pi = \{(\pi(x),x)\} \subset M_{red}^- \times \mu^{-1}(c) \subset M_{red}^- \times M$.)

<u>Conversely:</u> $L \subset (M,\omega)$ G-invariant Lagrangian $\Rightarrow \exists c \,/\, L \subset \mu^{-1}(c)$, hence L comes from above construction
(but may live in smooth part of a singular level of μ).

<u>Pf:</u> $\forall \xi \in \mathfrak{g}$, $X_\xi \in TL$ by invariance,
so $\forall v \in TL$, $\langle d\mu(v), \xi\rangle = \omega(X_\xi, v) = 0$ (L Lagrangian).
$\forall \xi \in \mathfrak{g}$
Holds $\forall \xi \in \mathfrak{g}$, hence $d\mu(v) = 0$ $\forall v \in TL$, ie. $\mu_{|L}$ constant.
(& central by equivariance). △

<u>Examples</u> ★ $(M^{2n},\omega) \circlearrowleft T^n$ toric variety, $c \in$ interior of Δ $\Rightarrow \mu^{-1}(c) \simeq T^n$ (orbit)
$\downarrow \pi$
$M_{red} = pt$.
T^n orbit $= \pi^{-1}(pt)$ is Lagrangian ✓.

★ $S^2 \times S^2 \circlearrowleft SO(3)$ $\quad \mu(v_1,v_2) = v_1 + v_2 \in \mathbb{R}^3 = so(3)^*$.
equal areas
$\mu^{-1}(0) =$ antidiagonal $\{(v,-v)\}$ is smooth, even though $SO(3)$ doesn't act freely (!)
It's Lagrangian. (doesn't quite fit exactly into above story...)

★ $\mathbb{C}^2 \times (\mathbb{CP}^1)^- \supset \{((z_1,z_2),(z_1:z_2)) \,/\, |z_1|^2 + |z_2|^2 = 1\} \simeq S^3$ graph of Hopf map
$\mathbb{C}^2 \supset S^3 \to S^2 = \mathbb{CP}^1$.
Lag. because: - explicit calc.^{n}.
- it's the Lag. correspondence underlying reduction for diag. S^1-actions on $\mathbb{C}^2 \supset \mu^{-1}(c) = S^3 \to M_{red} = \mathbb{CP}^1$

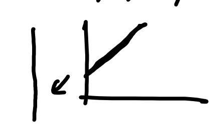
- or: S^1 acts on $\mathbb{C}^2\times(\mathbb{CP}^1)^-$ by diag. action on $\mathbb{C}^2$ (trivial on $\mathbb{CP}^1$), $M_{red}=\mathbb{CP}^1\times(\mathbb{CP}^1)^-\supset\Delta$ diagonal
  lift to $\pi^{-1}(\Delta)\subset\mu^{-1}(1)\simeq S^3\times\mathbb{CP}^1\subset\mathbb{C}^2\times\mathbb{CP}^1$.

- or: $SU(2)$ acts on $\mathbb{C}^2$, $\mu=\big(\frac{1}{2}(|z_1|^2-|z_2|^2), \mathrm{Im}(z_1\bar{z}_2), \mathrm{Re}(z_1\bar{z}_2)\big)$.
  $\left\{\begin{pmatrix}a & -\bar{b}\\ b & \bar{a}\end{pmatrix}, |a|^2+|b|^2=1\right\}$ Lie alg. $\begin{pmatrix}i & \\ & -i\end{pmatrix}$, $\begin{pmatrix} & -1\\ 1 & \end{pmatrix}$, $\begin{pmatrix} & i\\ i & \end{pmatrix}$ Levels = diag. S^1's.

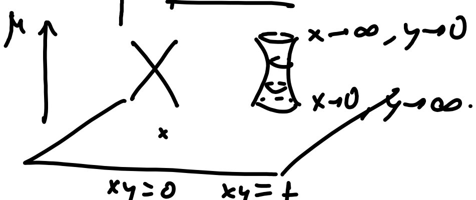
  and on $\mathbb{CP}^1$ (commutes with S^1-action, $\mu=\dfrac{\text{same}}{|z_1|^2+|z_2|^2}$ = embedding $\mathbb{P}^1\simeq S^2\subset\mathbb{R}^3$)

  so: $SU(2)$ action on $\mathbb{C}^2\times(\mathbb{CP}^1)^-$
  $\mu=\mu_{\mathbb{C}^2}-\mu_{\mathbb{CP}^1}$ $\quad\mu^{-1}(0)=S^3$ = graph of Hopf map
  = one orbit of $SU(2)$.

* Another ex. $(\mathbb{C}^2,\omega_0)\circlearrowleft S^1$ by $e^{i\theta}\cdot(x,y)=(e^{i\theta}x,e^{-i\theta}y)$, $\mu=\frac{1}{2}(|x|^2-|y|^2)$.
  For $c\neq 0$, $M_{red}\simeq\mathbb{C}$, either by toric geometry

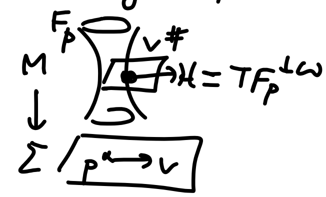


  or by explicit map: $\mathbb{C}^2\xrightarrow{\pi}\mathbb{C}$
  $(x,y)\mapsto xy$



  each fiber of π intersects $\mu^{-1}(c)$ in a single orbit of S^1-action
  So $\mu^{-1}(c)/S^1\xrightarrow[\text{diffeo}]{\sim}\mathbb{C}$ (with some area form, not the standard one!)
  Still true at $c=0$ except singularity at origin.
  So, now, given any simple closed curve $\gamma\subset\mathbb{C}$, $\pi^{-1}(\gamma)\cap\mu^{-1}(c)$ Lag. T^2.
  The Chekanov torus in $\mathbb{R}^4$ is the case of $c=0$, γ not enclosing origin.
  This also exists in $\mathbb{CP}^2$ or $S^2\times S^2$ (reduce wrt diagonal S^1-action ...).

* This is also an instance of: symplectic fibrations
  $\pi:(M,\omega)\to\Sigma^2$ with symplectic fibers ($\omega_{|\ker d\pi}$ non deg.), submersion out of crit pts.
  Horizontal distribution $\mathcal{H}=(\ker d\pi)^{\perp\omega}$ complement to fiber $F_p=\pi^{-1}(p)$
  Horizontal lift of a vector $v\in T\Sigma$: $v^\#\in\mathcal{H}$ s.t. $d\pi(v^\#)=v$.



  Fact: assuming fibers compact or properness, horizontal parallel transport induces symplectomorphisms between the fibers of π.
  (v vector field on Σ, $v^\#$ horiz lift, then $L_{v^\#}\omega=d\iota_{v^\#}\omega$ vanishes on TF).

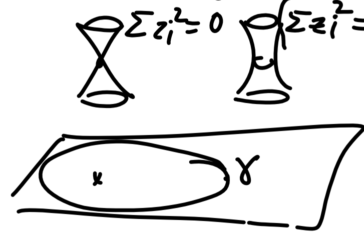
Now, $\ell\subset F_p=\pi^{-1}(p)$, γ curve through p, L = horiz parallel transport of ℓ over γ (i.e. flow by $\dot{\gamma}^\#$) is Lagrangian in M. Conversely, every Lag. $\subset M$ that fibers over an arc under π is of this form. In above example: parallel transport takes $\ell=S^1\subset\pi^{-1}(p)$  to S^1-orbit at same value of μ_{S^1}, and closes up to give T^2 over .

Instead of $T^2\subset\mathbb{C}^2$, can also build similarly $S^1\times S^{n-1}\subset\mathbb{C}^n$: ($S^{n-1}$-bundle$/S^1$)

$(z_1,\dots,z_n)\in\mathbb{C}^n$

$\downarrow\pi$

$z_1^2+\dots+z_n^2\in\mathbb{C}$



$c\in\mathbb{R}_+$: $\ell=F\cap\mathbb{R}^n$
$=\{(x_1..x_n)\in\mathbb{R}^n/\sum x_i^2=c\}$

(Lag. S^{n-1}, $\omega=\sum dx_i\wedge dy_i=0$ ✓)

For $\arg(c)=\theta$, $\ell=F\cap(e^{i\theta/2}\mathbb{R})^n$

These transport to each other.

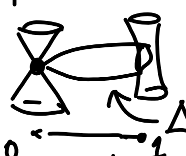
If γ doesn't enclose 0, get $L\simeq S^1\times S^{n-1}$.

If γ around 0, then $L\simeq S^1\times S^{n-1}$ if n even,
non-orientable S^{n-1}-bundle over S^1 if n odd.

(Note: fiber $F=\{\sum z_i^2=c\}$ has an antisympl. involution $(z_1..z_n)\mapsto e^{2i\theta}(\bar z_1,\dots,\bar z_n)$, $\arg(c)=\theta$, fixed point locus is automatically isotropic. ($\Rightarrow\ell$ Lag.) In fact F is symplectomorphic to (T^*S^{n-1},ω_0))

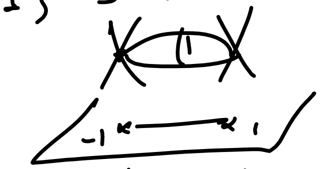
* In fact, this example is a prototype of a <u>sympl. Lefschetz fibration</u>
  = sympl. fibration with isolated critical points modelled on $(z_1..z_n)\mapsto\sum z_i^2$

  Then parallel transport def^d outside of critical points; set of pts $\to$ crit pt under parallel transport = <u>vanishing cycle</u> = Lagrangian S^{n-1} in the fiber



fibered over $\mathbb{R}_{\geq 0}\subset\mathbb{C}$

Can assemble these, e.g. $\{\sum_{i=1}^{n+1}z_i^2=1\}$ $\downarrow$ $\mathbb{C}$ $\quad S^n=\{(x_1..x_{n+1})\in\mathbb{R}^{n+1}/\sum x_i^2=1\}$



Δ Lefschetz thimble = Lag. disc $\subset M$
$\partial\Delta$ = vanishing cycle $\subset F$

"matching path" construction of Lag. spheres (Seidel)

* Monodromy of parallel transport around a singular fiber is Ham. isotopic to <u>Dehn twist</u> about the vanishing cycle (cf. HW)

* Every compact sympl. mfld carries a sympl. Lefschetz fibration outside a codim$_{\mathbb{R}}$ 4 locus (base locus of pencil, extend after blowup) [Donaldson 1998]

  Every compact Lagrangian can be made to fiber over a path in a suff. complicated sympl. L. fibration (A–Muñoz–Presas 2006). This is not effective in practice.

---

* <u>Generating functions</u>.

construct exact Lagrangians in $(T^*N,\omega=d\lambda)$ more general than graph(dF)

given $F: \underset{x}{N}\times\underset{u}{\mathbb{R}^k}\to\mathbb{R}$, assumed non-deg. in u direction: where $\frac{\partial F}{\partial u}=0$, $\left(\frac{\partial^2F}{\partial x\,\partial u}\ \ \frac{\partial^2F}{\partial u^2}\right)$ full rank

$$L_F:=\{(x,\xi)\in T^*N\,/\,\exists u\in\mathbb{R}^k,\ \frac{\partial F}{\partial u}(x,u)=0,\ \frac{\partial F}{\partial x}(x,u)=\xi\}$$

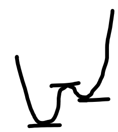
This is the image of $\mathrm{graph}(dF) \subset \underset{x,\xi}{T^*N} \times \underset{u,v}{T^*\mathbb{R}^k}$ under intersection with coisotropic $\{v=0\}$
($\cap$ is transverse by nondeg assumption)

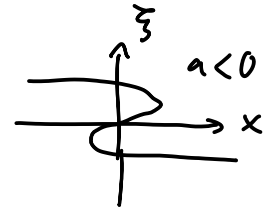
and projection to T^*N. Equivalently, reduction for $\mathbb{R}^k$-action by u-translation, moment map $= v$.

L_F is an immersed exact Lagrangian in T^*N (embedded if projection is injective)
$i^*\lambda = dF$ ↙ ↳ by reduction

<u>Ex</u>: $N=\mathbb{R}$, $\mathbb{R}^k=\mathbb{R}$

$F(x,u) = f(x) + u^2$ gives usual $\mathrm{graph}(df)$.

$F(x,u) = \frac{1}{4}u^4 + \frac{a}{2}u^2 + ux$ gives 



$\frac{\partial F}{\partial u} = 0 \Leftrightarrow u^3 + au + x = 0$, and $\xi = \frac{\partial F}{\partial x} = u$, so $\xi^3 + a\xi + x = 0$

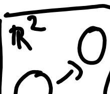
(graph for $a>0$, not $a<0$)

<u>Thm</u> (Laudenbach-Sikorav) 1985 ‖ $L \subset T^*N$ Ham. isotopic to the zero section
$\Rightarrow L$ has a generating function which is quadratic at infinity.

This leads to a proof of <u>Arnold's conj. for Lag. intersections</u> in cotangent bundles:

‖ N closed, $L \subset T^*N$ Hamiltonian isotopic to the zero section 0_N, $L \pitchfork 0_N$
$\Rightarrow \#(L \cap 0_N) \geq \sum \dim H^*(N)$.

For $\mathrm{graph}(df)$, f Morse, this is just the Morse inequality ($\#\,\mathrm{crit}(f) = \mathrm{rk}\, CM^*(f) \geq \mathrm{rk}\, HM^*(f) = \mathrm{rk}\, H^*(N)$).
For $L = L_F$ it's the same! $L \cap 0_N =$ crit pts of F, and Morse ineqs. hold
for functions on $N \times \mathbb{R}^k$ which are quadratic at infinity.

By Weinstein, this also gives Arnold for C^0-small Ham. isotopies of $L \subset (M,\omega)$. ⚠ false in general 

Lag. Floer homology is the main tool used to prove Arnold conj when holds (eg. exact Lagrangians)
and understand the role of holomorphic discs in its potential failure (eg when Lag. is displaceable).

- <u>Gromov's h-principle</u> for Lag. immersions (resp. isotropic embeddings)

<u>Def</u>: ‖ A <u>formal Lagrangian immersion</u> of L into (M,ω) is the data of
$$\begin{array}{ccc} F: TL & \longrightarrow & TM \\ \downarrow & \circlearrowright & \downarrow \\ f: L & \longrightarrow & M \end{array}$$
(ie F is a vect. bundle hom. $TL \to f^*TM$ over L)
s.t. (1) F injective on every fiber of TL
(2) $\forall p \in L$, $F(T_pL) \subset T_{f(p)}M$ is a Lagrangian subspace } ie: TL embeds as a Lagrangian subbundle in the sympl. vect. bundle $f^*TM \to L$.
(3) $[f^*\omega] = 0 \in H^2(L,\mathbb{R})$

Given a Lag. immersion, $F = df$ satisfies these; but in general F need not match df, in fact f need not be an immersion – or Lagrangian!

Thm (Gromov, Lees): Every formal Lag. immersion is homotopic among formal Lag. immersions to a Lag. immersion.

Eg: for $M=\mathbb{C}^n$, (3) holds, and f^*TM is trivial $\simeq \underline{\mathbb{C}}^n$. So (1)+(2) $\Leftrightarrow$ $\exists$ isomorphism of complex v.b. $TL\otimes\mathbb{C}\simeq\underline{\mathbb{C}}^n$. So: $\exists$ Lag. immersion $i: L \looparrowright (\mathbb{C}^n,\omega_0)$ iff $TL\otimes\mathbb{C}$ is trivial as cx. vector bundle.

Ex: $\exists$ Lag. immersion $S^n \looparrowright (\mathbb{C}^n,\omega_0)$ with just one transverse double point.
Comes from 

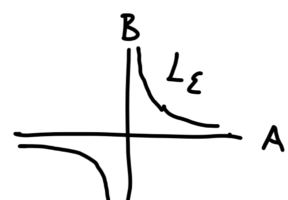
In fact, every closed oriented 3-mfld has a Lag. immersion in $\mathbb{C}^3$ with just one double pt.
On the other hand, no h-principle for Lag. embeddings; holom. curves / Floer theory give obstructions.
eg. for S^n, Thm (Gromov): $H_1(L)=0 \Rightarrow \nexists$ Lag. embedding $L\to\mathbb{C}^n$
(more generally $\nexists$ exact Lagrangians $\hookrightarrow\mathbb{C}^n$. Pf: L exact $\Rightarrow HF(L,L)=H^*(L)$, but displaceable $\Rightarrow HF=0$)

* To produce embeddings out of immersions: two constructions:

1) stabilization (Audin-Lalonde-Polterovich): $L \overset{i}{\looparrowright} M$ Lag. immersion $\Rightarrow$ $\exists$ Lag. embedding $L\times S^1 \hookrightarrow M\times\mathbb{R}^2$ close to $i\times$ inclusion.

(cf. homework; or: deform $i: L\looparrowright M\times\{0\}\subset M\times\mathbb{R}^2$ to an isotropic immersion, whose normal bundle is $\simeq (T^*L\times\mathbb{R}^2, 0\oplus\omega_0)$. Then use S^1's in $\mathbb{R}^2$ factor).

2) Lagrangian surgery (Polterovich).



* Model: $A=\mathbb{R}^n_{x_1\dots x_n}\times\{0\}$, $B=\{0\}\times\mathbb{R}^n_{y_1\dots y_n}\subset(\mathbb{R}^{2n},\omega_0)$.

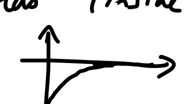
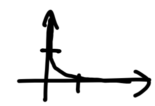
Consider $f_\varepsilon:\mathbb{R}^n-\{0\}\to\mathbb{R}$, $f_\varepsilon(x)=\varepsilon\log(\|x\|)$ for $\varepsilon>0$

then $L_\varepsilon := \mathrm{graph}(df_\varepsilon) = \left\{\left(x_1\dots x_n,\ y_1=\frac{\varepsilon x_1}{\|x\|^2},\ \dots,\ y_n=\frac{\varepsilon x_n}{\|x\|^2}\right)\right\}$ ie. $y=\frac{\varepsilon x}{\|x\|^2}$.

(conversely can take graph of $df_{-\varepsilon}$ over y-plane! ($x=\frac{\varepsilon y}{\|y\|^2}$ same))

This is topologically $\simeq\mathbb{R}^n-\{0\}\simeq\mathbb{R}\times S^{n-1}$, neck of size $\sqrt{\varepsilon}$, asymptotic to $A\cup B$ at infinity.

* To import this into arbitrary sympl. mflds (inside Darboux chart), use cut-off functions:

replace $f_\varepsilon(\|x\|)$ by cutoff like  whose derivative is 

Normalize so area under graph is equal to ε (or const. ε).

★ A, B Lagrangian $\subset (M,\omega)$, $A \cap B = \{p\}$ transverse intersection

$\Rightarrow$ define $B \#_p A$ in this way. Lag. embedding of connected sum $B \# A$

<u>Fact</u>: indep't of choices (local coord. chart, profile of f_ε, scaling param. $\varepsilon > 0$) up to Hamiltonian isotopy

<u>Significance (later)</u>: in Fukaya category, $A \#_p B \simeq \mathrm{Cone}(A \xrightarrow{p} B)$

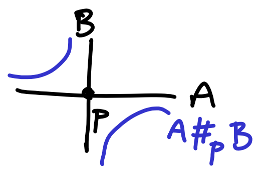
i.e. $\forall L\ \exists$ l.e.s. $HF(L,A) \xrightarrow{\cdot p} HF(L,B) \to HF(L, A\#_p B) \to \dots$

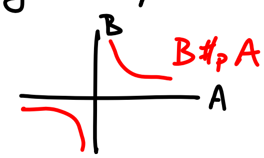
(Fukaya-Oh-Ohta-Ono)

(also Seidel if A is a sphere).

<u>However</u>: $A \#_p B$ is <u>not</u> Ham. iso (or in general smoothly iso.) to $B \#_p A$.

This is because in local model $x \leftrightarrow y$ corresp. to $\varepsilon \leftrightarrow -\varepsilon$.

In dim. 1: 

vs. 

$u = x_1 + ix_2$, $v = y_1 + iy_2$

(in dim. 2 only, they're Lag. isotopic; via hyperkähler geom. $L_\varepsilon \sim \{uv = \varepsilon\}$, $\varepsilon \in \mathbb{C}^* \checkmark$)

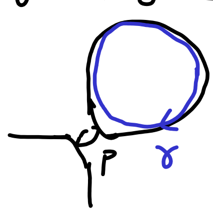
★ If two Lagrangians intersect in ≥ 2 points, or if L_0 self-intersects transversely, can perform surgeries at all intersections. Can choose direction of surgery (sign of ε) at each separately, and in fact can use different neck sizes...

$\to$ outcome of surgery may be a non-orientable Lagrangian! (depends on whether $\exists$ consistent orientation so that intersection numbers all $+1$)

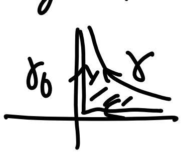
If dim. n is odd, can always choose surgery directions so the result is orientable.

<u>Ex</u>: Starting from $T^2 \subset \mathbb{R}^4$, deform to create self-$\cap$'s (even #, both signs) then surger these $\Rightarrow$ non-orientable surfaces with $4 | \chi$, $\chi < 0$. (Givental)

$\to$ Ham. isotopy class of surgery <u>does</u> depend on neck sizes in this case.

Eg. self-sum: 

neck S^{n-1} doesn't sweep any ω-area as ε varies (eg. because always bounds Lag. disc $\{y = \bar{x},\ |x| < \sqrt{\varepsilon}\}$ in local model. Anyway for $n-1 \geq 2$ there's no area to sweep!)

On the other hand, γ compared to γ_0 

differs by sympl. area $= c \cdot \varepsilon$

similarly  sweeps $\varepsilon_1 - \varepsilon_2$.